On Evaluating $\int_{-\infty}^{+\infty} e^{ax(x-2b)}dx$ by Contour Integration Round a Parallelogram

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Introduction. The usefulness of contour integration for the evaluation of infinite integrals can hardly be over-stressed. Strangely, however, and this can be something of an anti-climax for the student, the method finds no ready application to what is arguably the most useful and commonplace such integral, *viz.*, the probability integral

$$\int_{-\infty}^{+\infty} e^{-x^2}\, dx = \sqrt{\pi}. \tag{1}$$

Indeed it was widely held at one time that contour integration could not achieve this evaluation. By now, however, it is fairly well known that the probability integral can indeed be evaluated by a contour integration of a suitable integrand around a suitable skew (*i.e.*, non-rectangular) parallelogram. Such a proof was given by Mirsky [**3**] as long ago as 1949. There is an account of such an evaluation in [**2**]. It will be seen from either reference that neither the integrand nor the parallelogram contour employed is at all evident.

The related Fresnel integrals

$$\int_{-\infty}^{+\infty} \cos x^2\, dx = \sqrt{\frac{\pi}{2}} = \int_{-\infty}^{+\infty} \sin x^2\, dx,$$

which can be written in complex form as

$$\int_{-\infty}^{+\infty} e^{-ix^2}\, dx = \sqrt{\frac{\pi}{2}}\,(1-i), \tag{2}$$

are easier to handle and can be evaluated by integration of a suitable integrand round a suitable standard rectangle [**1**]. Again, however, the integrand in particular is not at all evident.

For the student familiar with the notion that integration around rectangles, and so by extension around parallelograms, is a useful technique, the difficulty still remains, regarding the probability and Fresnel integrals in particular, of finding suitable integrands and contours in order to make the desired evaluations. It hardly satisfies, even if it suffices, to be bidden without explanation to integrate this function round that parallelogram. And even if one is, the doubt lingers whether it is really necessary to have to resort to a skew parallelogram for the probability integral when a rectangle serves so well for the Fresnel integrals. The purpose here is to shed light on these matters but in relation to the more general integral

$$\int_{-\infty}^{+\infty} e^{ax(x-2b)}dx, \tag{3}$$

with $a, b \in \mathbb{C}$ suitably restricted for convergence, of which the probability and Fresnel integrals are the respective special cases $b := 0, a := -1, -i$. We analyse

the two cases

i) re $a < 0$, into which the probability integral (1) falls, and
ii) re $a = 0 >$ im a and im $b \le 0$, into which the Fresnel integral (2) falls.

It transpires that, in these cases, the integral (3) evaluates to

$$\pm i \frac{\sqrt{\pi}}{\sqrt{a}} e^{-ab^2},$$

wherein $\sqrt{a}$ is the principal square root of a and the sign is fixed by a definite rule. In itself this evaluation is a secondary issue; being a rather routine, if technical, matter once an integrand and contour have been specified. There are, moreover, other evaluations for the probability and Fresnel integrals easier than the contour integrations proposed here. The primary aim here is to demonstrate to the student, as regards the probability and Fresnel integrals in particular, that an evaluation by contour integration is possible, how an integrand and parallelogram contour suitable for the evaluation can be hit upon and to show why, given the tack taken, rectangular contours cannot work for the probability integral in particular.

A third case, *viz.*, *iii*) re $a = 0 <$ im a and im $b \ge 0$, can be reduced to the case *ii*) by conjugation, on noting that

$$\overline{\int_{-\infty}^{+\infty} e^{ax(x-2b)}\,dx} = \int_{-\infty}^{+\infty} e^{\bar{a}x(x-2\bar{b})}\,dx,$$

when either integral exists. By this ruse of conjugation we may, and hereafter do, assume that im $a < 0$ when re $a = 0$.

Throughout, for any $z \in \mathbb{C}$, $\sqrt{z}$ denotes the (principal) square root of z, *viz.*, $\sqrt{r}e^{i\theta/2}$ when $z =: re^{i\theta}$ with $r \ge 0$, $-\pi < \theta \le \pi$. The square roots of z are then precisely $\pm\sqrt{z}$.

Determination of integrand and parallelogram. Let $a, b \in \mathbb{C}$ with $a \ne 0$. For any R, with $0 < R \le +\infty$, let

$$I(R) := \int_{-R}^{R} e^{ax(x-2b)}\,dx.$$

In case $R \ne +\infty$, let γ be the parallelogram $ABCD$ with vertices $\pm\alpha \pm R\beta$ as shown in Figure 1, where $\alpha \in \mathbb{R}$, with $\alpha > 0$ and $\beta \in \mathbb{C}$, with $0 < \arg\beta < \pi$.

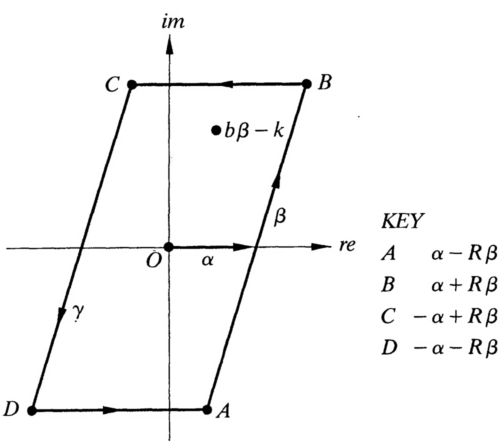

Figure 1. Parallelogram contour γ.

We consider the contour integral $I := \int_\gamma f(z)\,dz$, wherein f and γ are to be chosen so as to permit evaluation of $\int_{-\infty}^{+\infty} e^{ax(x-2b)}\,dx = I(+\infty)$, as the Cauchy principal value $\lim_{R\to+\infty} I(R)$.

Towards the evaluation of I, integration along the sides AB and CD of γ, contributes

$$\beta \int_{-R}^{+R} [f(x\beta+\alpha) - f(x\beta-\alpha)]\,dx$$

which is $I(R)$ for the choice $\beta[f(x\beta+\alpha) - f(x\beta-\alpha)] = \exp ax(x-2b)$. On writing $z = x\beta + \alpha$,

$$f(z) - f(z-2\alpha) = \frac{1}{\beta}\exp a\left(\frac{z-\alpha}{\beta}\right)\left(\frac{z-\alpha}{\beta} - 2b\right) =: h(z),$$

so that

$$h(z) = \frac{1}{\beta}\exp\frac{a}{\beta^2}(z-\alpha)(z-\alpha-2b\beta) \tag{4}$$

$$= h(0)\exp\frac{a}{\beta^2}z(z-2\alpha-2b\beta). \tag{5}$$

This suggests that a suitable integrand might be got by solving the equation $f(z) - f(z-2\alpha) = h(z)$ for $f(z)$ and to that end we have the following proposition, whose proof is by direct substitution.

Proposition 1. *Suppose that $h(z-2\alpha) = h(z)g(z)$ for some function g with period 2α and each $z \in \mathbb{C}$. Then*

$$f(z) = \frac{h(z)}{1-g(z)}$$

solves $f(z) - f(z-2\alpha) = h(z)$, when $g(z) \neq 1$.

We now seek to determine a suitable function g. From (4) and (5),

$$h(z-2\alpha) = h(0)\exp\frac{a}{\beta^2}(z-2\alpha)(z-4\alpha-2b\beta)$$

$$= h(z)\exp\frac{8\alpha^2 a}{\beta^2}\exp\frac{4\alpha a}{\beta^2}(b\beta - z).$$

Thus

$$\exp\frac{8\alpha^2 a}{\beta^2}\exp\frac{4\alpha a}{\beta^2}(b\beta - z)$$

serves as a suitable g provided it has period 2α, thus provided $2\alpha = 2\pi i\beta^2/4\alpha a$ or $4\alpha^2 a = \pi i \beta^2$. Since this equation merely determines the ratio β/α, we may (at the expense of replacing β by $\beta/2\alpha$) assume that $\alpha := 1/2$. With that choice, $a = \pi i \beta^2$ so that

$$\beta := \pm\frac{\sqrt{a}}{\varpi\sqrt{\pi}}, \tag{6}$$

where $\varpi := \sqrt{i} = e^{i\pi/4}$ and

$$\text{the sign is chosen to ensure that } 0 < \arg\beta < \pi. \tag{7}$$

Note from the equation $a = \pi i \beta^2$ that β is real if and only if $\operatorname{re} a = 0 < \operatorname{im} a$. Thus β is not real, by assumption already made that $\operatorname{im} a < 0$ when $\operatorname{re} a = 0$. Then with these choices for α and β,

$$\begin{aligned} h(z) &:= \frac{1}{\beta}\exp \pi i(z - 1/2)(z - 1/2 - 2b\beta) \\ &= h(0)\exp \pi i z(z - 1 - 2b\beta), \\ g(z) &:= \exp 2\pi i(b\beta - z), \quad \text{and} \\ f(z) &:= \frac{h(z)}{1 - g(z)}. \end{aligned} \tag{8}$$

It remains to verify that these choices yield the required evaluation.

Before that, however, observe that the contour chosen is rectangular if and only if β takes the form $i\lambda$ for some $\lambda > 0$. From the equation $a = \pi i \beta^2$, that is the case if and only if $\operatorname{re} a = 0 > \operatorname{im} a$; a condition satisfied by the Fresnel integral but not the probability integral. Thus, given this approach, a skew parallelogram contour is inevitable for evaluation of the probability integral.

Evaluation of the integral. Clearly the singularities of f are simple poles at $z = b\beta$ (mod 1), where $g(z) = 1$. [We leave aside the case when one of these poles falls on the side AB of γ, so that a second falls on the side CD. This is the case, for large R at least, precisely when $2\operatorname{im} b$ is an odd integral multiple of $\operatorname{im}(1/\beta)$. Then we must make small indentations in the usual way or make some other provision in the analysis, *e.g.*, by shifting γ half a unit to the right.] Outside that exceptional case, just one of the poles, say $b\beta - k$ for some integer k, is inside the parallelogram when $R > |b|$. Its residue there is

$$\frac{h(b\beta - k)}{-g'(b\beta - k)} = \frac{1}{2\pi i}\frac{h(b\beta - k)}{g(b\beta - k)} = \frac{1}{2\pi i}h(b\beta - k),$$

since $g(b\beta - n) = 1$ for each $n \in \mathbb{Z}$. Further, since $h(z-1) = h(z)g(z)$ for each $z \in \mathbb{C}$, it follows by easy inductions that $h(b\beta - n) = h(b\beta)$ for each $n \in \mathbb{Z}$. Furthermore from (6) and (8), one checks that $h(b\beta) = \varpi\, e^{-ab^2}/\beta$. Hence by the residue theorem,

$$I = 2\pi\frac{1}{2\pi i}h(b\beta - k) = h(b\beta) = \frac{\varpi}{\beta}\, e^{-ab^2}.$$

On the other hand,

$$\begin{aligned} I &= \left(\int_{\overrightarrow{AB}} + \int_{\overrightarrow{CD}}\right) f(z)\,dz + \left(\int_{\overrightarrow{BC}} + \int_{\overrightarrow{DA}}\right) f(z)\,dz \\ &= I(R) + \int_{-1/2}^{1/2} f(R\beta - x)\,dx + \int_{-1/2}^{1/2} f(-R\beta + x)\,dx \\ &=: I(R) + J(R) + K(R), \quad \text{say.} \end{aligned}$$

We shall show that, for suitable restriction upon a and b, $J(R) \to 0$ as $R \to +\infty$. A similar analysis shows likewise that $K(R) \to 0$ as $R \to +\infty$. Thus, proceeding to the limit as $R \to +\infty$, we conclude that

$$\int_{-\infty}^{+\infty} e^{ax(x-2b)}\,dx = \lim_{R\to+\infty} I(R) = I = \frac{\varpi}{\beta}e^{-ab^2} = \pm i\frac{\sqrt{\pi}}{a}e^{-ab^2}, \tag{9}$$

with the sign chosen as in (7). This is the desired evaluation.

It remains to verify that $J(R) \to 0$ as $R \to +\infty$.

$$|J(R)| = \left|\int_{-1/2}^{1/2} f(R\beta - x)\,dx\right| = \left|\int_{-1/2}^{1/2} \frac{h(R\beta - x)}{1 - \exp 2\pi i[(b+R)\beta - x]}\,dx\right|$$

$$\leq \int_{-1/2}^{1/2} \frac{|h(R\beta - x)|}{|1 - e^{w(x)}|}\,dx, \tag{10}$$

where for $|x| \leq 1/2$,

$$w(x) := 2\pi i[(b+R)\beta - x] = 2\pi(b+R)\beta i - 2\pi x i. \tag{11}$$

In order to proceed with the estimate, we use the

Proposition 2. *For any* $w =: u + iv \in \mathbb{C}$, $|1 - e^w| \geq 1/2$ *when* $|u|$ *is sufficiently large.*

Proof: $|1 - e^w| \geq |1 - |e^w|| = |1 - e^u| \to \begin{cases} +\infty & \text{as} \quad u \to +\infty, \\ 1 & \text{as} \quad u \to -\infty. \end{cases}$ ∎

Now from (11), as $R \to +\infty$,

$$|u(x)| := |\mathrm{re}\, w(x)| = 2\pi|\mathrm{im}\,[(b+R)\beta]| \to +\infty.$$

Note that this occurs independently of x. Thus for all sufficiently large R, independently of x, $|1 - e^{w(x)}| \geq 1/2$ for $|x| \leq 1/2$. Hence from (10), for all sufficiently large R,

$$|J(R)| \leq 2\int_{-1/2}^{1/2} |h(R\beta - x)|\,dx$$

$$= 2|h(0)|\int_{-1/2}^{1/2} |\exp \pi i(R\beta - x)(R\beta - x - 1 - 2b\beta)|\,dx$$

$$=: 2|h(0)|\int_{-1/2}^{1/2} e^{\phi(x)}\,dx, \tag{12}$$

where

$$\phi(x) := \mathrm{re}\,[\pi i(R\beta - x)(R\beta - x - 1 - 2b\beta)]$$
$$=: (\mathrm{re}\, a)R^2 + c(x)R + d(x), \tag{13}$$

for some real-valued continuous functions c, d. Thus for $|x| \leq 1/2$,

$$\phi(x) \leq (\mathrm{re}\, a)R^2 + CR + D =: \varphi(R), \quad \text{say},$$

where $C := \max_{|x| \leq 1/2} c(x)$ and $D := \max_{|x| \leq 1/2} d(x)$.

Case i): $\mathrm{re}\, a < 0$. As $R \to +\infty$, $\varphi(R) \to -\infty$. Further from (12), for all sufficiently large R,

$$|J(R)| \leq 2|h(0)|\int_{-1/2}^{1/2} e^{\phi(x)}\,dx \leq 2|h(0)|e^{\varphi(R)}.$$

Thus $J(R) \to 0$ as $R \to +\infty$. ∎

Case ii): $\mathrm{re}\, a = 0 > \mathrm{im}\, a$ and $\mathrm{im}\, b \leq 0$. Then $a =: -iA$, for some $A > 0$. By choice of β, $-iA = a = \pi i \beta^2$, and $\arg \beta > 0$, so that

$$\beta = i\frac{\sqrt{A}}{\sqrt{\pi}}. \tag{14}$$

Now for $|x| \le 1/2$, from (13) with re $a = 0$, $\phi(x) = c(x)R + d(x) \le c(x)R + D$, where

$$\begin{aligned} c(x) &= \text{re}\left[-\pi i(2\beta x + \beta + 2b\beta^2)\right] = \text{re}\left[-\pi i\beta(2x+1) - 2\pi\beta^2 ib\right] \\ &= -\sqrt{A\pi}(2x+1) + 2A\,\text{im}\,b, \quad \text{from (14)}. \end{aligned}$$

From (12), for all sufficiently large R,

$$\begin{aligned} |J(R)| &\le 2|h(0)|\int_{-1/2}^{1/2} e^{\phi(x)}\,dx \\ &\le |h(0)|\,e^{(2AR\,\text{im}\,b)+D}\int_{-1/2}^{1/2} e^{-\sqrt{A\pi}R(2x+1)}\,dx \\ &= |h(0)|\,e^{(2AR\,\text{im}\,b)+D}\,\frac{\left(1-e^{-2\sqrt{A\pi}R}\right)}{2\sqrt{A\pi}R} = K\left(1-e^{-2\sqrt{A\pi}R}\right)\frac{e^{2AR\,\text{im}\,b}}{R}, \end{aligned}$$

where $K := |h(0)|e^D/2\sqrt{A\pi}$. Both $e^{-2\sqrt{A\pi}R}$ and $e^{2AR\,\text{im}\,b}/R$ tend to 0 as $R \to +\infty$. Thus $J(R) \to 0$ as $R \to +\infty$. ■

In this second case note, from (9), that

$$\int_{-\infty}^{+\infty} e^{ax(x-2b)}\,dx = \frac{\varpi}{\beta}e^{-ab^2} = -i\varpi\frac{\sqrt{\pi}}{\sqrt{A}}e^{iAb^2}.$$

Evaluation of the probability and Fresnel integrals. Let $b := 0$. The respective particular choices $a := -1$ and $a := -i$ yield from (6)

$$\beta = \pm\frac{\sqrt{-1}}{\varpi\sqrt{\pi}} = \frac{\varpi}{\sqrt{\pi}} \quad \text{and} \quad \beta = \pm\frac{\sqrt{-i}}{\varpi\sqrt{\pi}} = \frac{i}{\sqrt{\pi}},$$

on choosing the sign to ensure that arg $\beta > 0$ in each case. These yield respectively, from (9),

- the probability integral $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \frac{\varpi}{\beta}e^0 = \sqrt{\pi}$, and
- the Fresnel integral $\int_{-\infty}^{+\infty} e^{-ix^2}\,dx = \frac{\varpi}{\beta}e^0 = -i\varpi\sqrt{\pi} = \sqrt{\frac{\pi}{2}}\,(1-i)$.

Note that, in the case of the probability integral, the parallelogram contour is skewed at an angle of $\pi/4$ from the vertical. In the cases of the Fresnel integral the contour is a rectangle in standard position.

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